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1990 J. Phys. A: Math. Gen. 23 L871

(http://iopscience.iop.org/0305-4470/23/17/004)

J. Phys. A: Math. Gen. **23** (1990) L871–L875. Printed in the UK

LETTER TO THE EDITOR

Common origin of different bifurcation pictures in Yang–Mills field equations with sources

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Received 24 May 1990, in final form 18 June 1990

Abstract. We argue that different bifurcation pictures of $\xi(Q)$ curves can be regarded as a manifestation of the different ways in which the total energy $\xi(\lambda)$ and the total external charge $Q(\lambda)$ vary with their parameter λ near their respective minimum values. From this point of view we rederive, in a simple way, certain well known results.

The bifurcation phenomenon, a nonlinear effect, occurs in many areas of physics and it certainly plays a role in particle physics [1]. For the classical Yang–Mills (YM) field equations with an external source, the term bifurcation normally refers to one of the following two cases:

(*a*) the bifurcation of the gauge field configuration as some parameter, say λ, is varied in the equations;

(*b*) the branching of the total energy $\xi(\lambda)$ of the gauge fields and external matter sources as the total external charge $Q(\lambda)$ is varied.

These two cases are related. Bifurcation as in case (*a*) above, at the critical value λ_c of the parameter, requires the existence of a zero-eigenvalue mode in the stability (fluctuation) equations for the gauge fields [2]. It is then argued that ξ and Q may possess a common minimum at $\lambda = \lambda_c$. In fact this is what was done by Jackiw and Rossi [2] who chose the parameter ε, which is the expansion parameter around the bifurcation point, to be positive. The existence of a common local minimum of ξ and Q does imply the bifurcation of $\xi(Q)$, and also of the gauge field with a suitably redefined parameter. In this letter we point out that the various bifurcation pictures discussed in the literature can simply be understood by examining their behaviour near the bifurcation point and that they have common characteristics. Finally we rederive in a simple way some standard known results by using small-parameter expansions about the bifurcation point.

In the literature, several different bifurcation pictures of $\xi(Q)$ have been exhibited.

(i) When the external charge density is specified in the radial gauge frame with the Kronecker index equal to one, it was first shown by Jackiw *et al* [3] that when the strength Q of a spherical shell source exceeds a non-zero critical value, the type-II solutions bifurcate into two branches with different total energy. See figure 1 of [3].

(ii) For the external source in an Abelian gauge frame with zero Kronecker index, the Abelian Coulomb solution becomes unstable at the bifurcation point from which a stable magnetic multipole solution emanates. See figure 2 of [4]. The bifurcation picture of [5] also belongs to this type.

(iii) For the type-II solutions discussed in [6], if the parameters are suitably constrained, ξ and Q do have their respective minima at the same critical value of the parameter. See figure 2 of [7]. Here one has only one branch in the ξ against Q diagram since the gauge field is on the verge of becoming imaginary at the critical parametric value.

(iv) Recently it was found [8] that one can in fact construct infinite pairs of bifurcating branches emanating from a bifurcation point and there can be an infinite number of bifurcation points. See figure 1 of [8].

These different pictures of bifurcation can be easily understood if we direct our attention to the behaviour of ξ and Q when the parameter λ is varied. (Note that λ can be a set of parameters.) The common important feature of the bifurcation pictures [3-8] is the cusp-like behaviour in the $\xi(Q)$ curve. The cusp arises when $d\xi/dQ$ does not exist at $\lambda=\lambda_c$ which in turn means the first derivatives of $\xi(\lambda_c)$ and $Q(\lambda_c)$ with respect to λ vanish. Thus it is the different ways in which $\xi(\lambda)$ and $Q(\lambda)$ change with λ in the vicinity of λ_c that gives rise to the different bifurcation pictures of the ξ against Q plot mentioned above. As long as either $\xi(\lambda)$ or $Q(\lambda)$ are not symmetrical with respect to their respective minima†, i.e. $\xi(\lambda_1)\neq\xi(\lambda_2)$ or $Q(\lambda_1)\neq Q(\lambda_2)$ where $\lambda_{1,2}=\lambda_c\pm\Delta\lambda$, for any small positive increment $\Delta\lambda$, then as λ is varied throughout its allowed range, one will obtain the two branches ending with a cusp, the same as that given by [3], i.e. the bifurcation picture (i). If the gauge field configuration can be labelled by several parameters, then different constraints may be set up between the parameters to yield different sets of $\xi(\lambda)$ and $Q(\lambda)$ so that one can engineer as many pairs of bifurcating branches as one wishes, thus arriving at the picture (iv) mentioned above. However, if both $\xi(\lambda)$ and $Q(\lambda)$ are symmetrical with respect to their respective minima, then the two branches in picture (i) will coalesce into one branch. Indeed picture (ii) can be obtained from picture (iv) in this way by demanding $\xi(\lambda_1)=\xi(\lambda_2)$ and $Q(\lambda_1)=Q(\lambda_2)$ for the respective two pairs in (iv). In [4] the parameter used is c and $\xi(-c)=\xi(c)$, $Q(-c)=Q(c)$ for a non-zero c. Of course the bifurcation picture of the Abelian Coulomb solutions has more meaning than merely a degenerate case of (iv) because it occurs naturally (no artificial constraints on the parameters) and the magnetic multipole solution co-exists with the Coulomb solution for the same charge density, not just same total charge. The bifurcation picture (iii) arises because the parameter λ is only allowed to vary from λ greater than λ_c, no solution exists for $\lambda<\lambda_c$. Consequently picture (iii) is not a degenerate case of (i).

In view of the above consideration, we may in principle employ a complicated enough charge density distribution and obtain solutions to produce $\xi(\lambda)$ and $Q(\lambda)$ with as many common local minima or maxima as we wish which then lead to the ξ against Q diagram with the corresponding number of cusps. This is sketched in figures 1 and 2. In short, provided $\xi(\lambda)$ and $Q(\lambda)$ are at least quadratic in their argument λ, and λ denotes a set of parameters, one can always manufacture a $\xi(Q)$ curve with two branches terminating at a cusp.

One useful outcome of the above viewpoint regarding the bifurcation of the $\xi(Q)$ diagram is that some standard known results can be derived in a simple manner as follows. Expanding $Q(\lambda)$ about λ_c yields

$$(Q-Q_c)=Q_2(\Delta\lambda)^2 \qquad Q_2=\frac{1}{2}\frac{d^2Q(\lambda_c)}{d\lambda^2} \qquad Q_c=Q(\lambda_c). \tag{1}$$

† We exclude the trivial case: $\xi(\lambda)$ and $Q(\lambda)$ have the same shapes, leading to a linear relation between them.

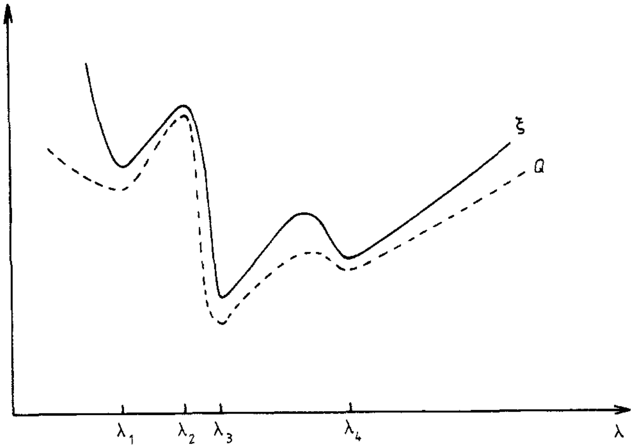

Figure 1. Variations of the total energy $\xi(\lambda)$ and total charge $Q(\lambda)$ with the parameter λ. Common local minima occur at λ_1, λ_3 and λ_4 respectively whilst a common local maximum is at λ_2. Note that there is no common maximum between λ_3 and λ_4.

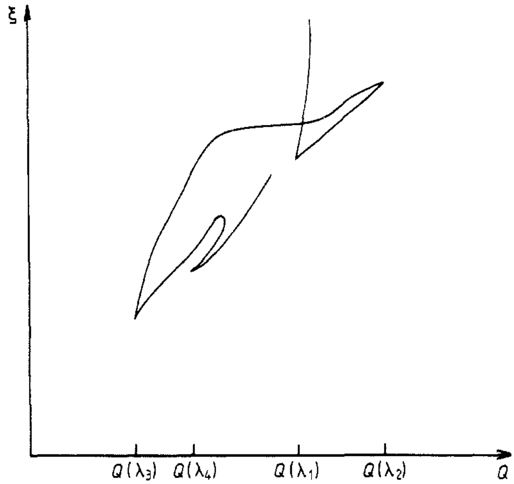

Figure 2. A sketch of the $\xi(Q)$ curve corresponding to the $\xi(\lambda)$ and $Q(\lambda)$ as given by figure 1.

Equation (1) shows that for $\Delta\lambda$ small, λ_1 and λ_2 give the same total charge and they correspond to the two points on the upper and lower branches of the $\xi(Q)$ curve. Hence,

$$\Delta\xi = |\xi(\lambda_1) - \xi(\lambda_2)| = 2|\xi_3(\Delta\lambda)^3|$$
$$\xi_3 \equiv \frac{1}{3!}\frac{d^2\xi(\lambda_c)}{d\lambda^3}. \tag{2}$$

Equations (1) and (2) together imply

$$\Delta\xi \simeq |2\xi_3 Q_2^{-3/2}|(Q-Q_c)^{3/2} \qquad Q > Q_c. \tag{3}$$

Similarly we obtain

$$\bar{\xi} = \tfrac{1}{2}|\xi(\lambda_1) + \xi(\lambda_2)| \simeq \xi_c + (\xi_2/Q_2)(Q-Q_c)$$
$$\xi_c \equiv \xi(\lambda_c). \tag{4}$$

Furthermore, if we let $a(\lambda)$ represent a typical gauge field (we have suppressed various indices), then again a Taylor expansion around the bifurcation point gives

$$a(\lambda) = a_c + a_1 \Delta\lambda \qquad a_c \equiv a(\lambda_c) \qquad a_1 \equiv \frac{\partial a(\lambda_c)}{\partial \lambda}.$$

Therefore

$$\Delta a = a(\lambda) - a(\lambda_c) \simeq \pm \frac{a_1}{Q_2^{1/2}} (Q - Q_c)^{1/2} \qquad Q > Q_c. \tag{5}$$

Note that in the above analysis it is essential that $Q_2 \neq 0$ and $\xi_3 \neq 0$. Equation (3) shows that the energy bifurcates as $(Q - Q_c)^{3/2}$ from the common minimum of ξ and Q. Equation (4) indicates that the mean energy of the two branches rises linearly with $(Q - Q_c)$ near the bifurcation point. Equation (5) implies that if we use $\mu = (Q - Q_c)$ as a new parameter, then the gauge fields bifurcate from $\mu = 0$. These results were first obtained by Jackiw and Rossi [2] using a much more involved analysis. The novel feature of our approach is that it demonstrates that the results are a simple mathematical consequence of two functions sharing a common local extremum point. In addition the present derivation provides computable coefficients in the expressions (3)-(5) which we have verified for a known analytic solution. We note that these results (3)-(5) are employed by the authors of [9] to discuss stability problems.

An important point to bear in mind is that although we have shown that bifurcation in $\xi(Q)$ (from a common local minimum of ξ and Q) also results in bifurcation of the fields, the later bifurcation is now described by the parameter μ and not λ which is the original parameter in the equations. In fact the two corresponding points on the bifurcating branches correspond to two different values of λ and therefore λ is not the bifurcation parameter in the present case. Finally the results (3), (4) and (5) are not applicable between the Coulomb solutions and magnetic multipole solutions, namely for the bifurcation picture (ii), since each family of these solutions is a degenerate case of the picture (i).

References

[1] Chang N P 1990 Bifurcation theory and dynamical symmetry breaking *First Asia-Pacific Workshop on High Energy Physics, June, 1987* (Singapore: World Scientific) to be published
Chang L N and Chang N P 1985 *Phys. Rev. Lett.* **54** 2407
Atkinson D and Johnson P W 1986 Bifurcation of the quark self-energy: infra-red and ultra-violet cut-offs *Preprint* Institute for Theoretical Physics, Groningen
Cicogna G 1981 *Lett. Nuovo Cimento* **31** 600
Woo C H 1981 *Nucl. Phys.* B **192** 125
Chow S N and Hale J K 1982 *Methods of Bifurcation Theory* (Berlin: Springer)
Chang S J and McCown J 1984 *Phys. Rev.* A **30** 1149
Steeb W H, Louw J A and Villet C M 1986 *Phys. Rev.* D **33** 1174
Matinyan S G, Prokhorenko E B and Savvidi G K 1986 *JETP Lett.* **44** 138
Malec E 1982 *J. Math. Phys.* **23** 21
Oh C H 1984 *J. Math. Phys.* **25** 660
[2] Jackiw R and Rossi P 1980 *Phys. Rev.* D **21** 426
[3] Jackiw R, Jacobs L and Rebbi C 1979 *Phys. Rev.* D **20** 474
[4] Oh C H, Chow S N and Lai C H 1984 *Phys. Rev.* D **30** 1334
[5] Paul S K and Khare A 1986 *Phys. Rev.* D **33** 1181
Paul S K and Khare A 1984 *Phys. Lett.* **138B** 402

[6] Oh C H, Teh R and Koo W K 1982 *Phys. Rev.* D **32** 3263
[7] Oh C H, Lai C H and Soo C P 1985 *Phys. Rev.* D **32** 2843
[8] Oh C H and Parwani R R 1988 *J. Math. Phys.* **29** 220
[9] Horvat D 1986 *Phys. Rev.* D **34** 1197
Carson L J 1984 *Phys. Rev.* D **29** 2355